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Anomalous scaling in depinning transitions

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It is demonstrated that the renormalization group (RG) flows of depinning transitions do not depend on whether the driving force or the system velocity is kept constant. This allows for a comparison between RG results and corresponding self-organized critical models. However, close to the critical point, scaling functions cross over to forms that can have singular behavior not seen in equilibrium thermal phase transitions. These can be different for the constant force and constant velocity driving modes, leading to different apparent critical exponents. This is illustrated by comparing extremal dynamics for interface depinning with RG results, deriving the change in apparent exponents. Thus, care has to be exercised in such comparisons.

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Driven dynamics of disordered systems have been studied extensively from two different approaches. As the external driving force is increased, a system undergoes a depinning transition from a macroscopically static state (with transient motion) to a moving steady state. This transition has been successfully described as a dynamical second order phase transition [1], and analyzed within the framework of the renormalization group (RG), originally for charge-density waves (CDWs) [2] and subsequently for interfaces [3] and a variety of other systems [4,5]. This approach obtains the scaling behavior near the transition where the system starts to move [6]. On the other hand, the concept of self-organized criticality (SOC), in the original sandpile model [7] and descendants thereof [8] has been used to obtain the behavior of systems that are forced to stay *at* the depinning transition.

One would expect there to be connections between these two viewpoints [9]; indeed, it is possible to map automaton models for CDWs to the original SOC model [5,10], and the 1+1 dimensional moving interfaces to a completely different SOC model [11]. The connections between the two have been exploited to obtain the dynamical critical exponent z for CDWs by borrowing from exact results for SOC sandpiles [12]. However, it has been pointed out [8] that one has to be careful whether the depinning transition is approached with a time-independent external driving force, tuned to its critical value, or with a time-independent (infinitesimal) velocity. In the latter case, one envisages a feedback loop that adjusts the external force in a time-dependent way, so as to ensure that the rate at which the system moves (averaged over its entire spatial extent) is strictly time independent. It has been suggested [8] that the critical behavior at the depinning transition could be different for these two driving modes. As an extreme example of constant current driving, there is the class of ''extremal models'' [13–15], where at every (discrete) time step there is activity only at one lattice site in the system.

For constant force driving, there are fluctuations in the (spatially averaged) velocity of the system, while for constant velocity driving there are fluctuations in the external force. In either case, at a nonzero driving rate, the fluctuations are small for sufficiently large systems, and thus the two driving mechanisms should be equivalent. However, the large system limit is problematic at the critical point. For instance, with constant force driving, the mean square fluctuations in the velocity of a large system are $(\delta v)^2 \sim v^2(\xi/L)^d$, where v is the mean velocity, ξ is the correlation length at velocity v (defined through the velocity autocorrelation function or other methods), L is the linear extent of the system, and d is its dimensionality. One would expect that holding the velocity constant by adjusting the force

would make a qualitative difference to the dynamics when the velocity fluctuations are a significant fraction of the mean velocity, i.e., $\xi \sim L$. Heuristically, for a fixed v and $L \rightarrow \infty$, imposing a constant velocity does not affect the dynamics, since the total ''activity'' at every time step is vL^d, so that avalanches, etc., are free to unfold with their own internal time scales. On the other hand, for fixed L as $v \rightarrow 0$, imposing a constant velocity ''chokes off'' the dynamics: avalanches proceed sequentially, with internal dynamics determined by the velocity constraint.

In this paper, we show that (to the extent that continuum descriptions are appropriate) both constant force and constant velocity driving actually have the *same* scaling under the renormalization group (for steady state behavior). However, scaling functions for various quantities, which can have different behaviors in the limits $\xi \ll L$ and $\xi \gg L$, are often different in the $\xi \gg L$ regime for the two driving modes. It is standard for scaling functions in equilibrium critical phenomena to behave differently when their arguments are small and large, but as will be discussed later, dynamical critical points have even more freedom in how scaling functions can behave. It is this freedom that allows identical RG flows to still yield different scaling functions for the two driving modes in the $\xi \gg L$ regime.

We demonstrate this by focusing on a particular system, that of a interface with short range internal elastic interactions that is driven through a disordered medium by an external force. Due to the disorder, different parts of the interface experience different random pinning forces as they move forward. Using the RG relationship between the interface velocity and the correlation length [3] $v \sim \xi^{\zeta - z}$, the crossover to $\xi > L$ occurs at $v \sim L^{\zeta - z}$. Note that extremal dynamics for a $(d+1)$-dimensional interface corresponds to $v \sim L^{-d}$, and $z - \zeta < d$ in all dimensions, so that $\xi \gg L$ for large L.

It has been pointed out [16,14,17] that in extremal models for the motion of pinned interfaces, the roughness of the interface can scale anomalously. Thus for a $(1+1)$-dimensional interface in a system of size L in steady state [18], if $h(x,t)$ is the interface position as a function of the transverse coordinate x and the time t, the roughness $w(t) = [\langle \{h(x,t+t_0) - \bar{h}(t+t_0) - h(x,t_0) + \bar{h}(t_0)\}^2 \rangle]^{1/2}$ (all averages are spatial averages) has the scaling form [17,19]

$$w(t) = t^{\beta_v} L^{-1/2} \varphi \left(\frac{L}{t^{1/z_v}} \right). \tag{1}$$

(The subscripts on β and z denote constant velocity driving.) It can be shown [14,17] that the scaling function φ goes to a constant for large values of its argument. The explicit L dependence that remains is anomalous, contrary to the normal expectation of a well defined $L \rightarrow \infty$ limit. As a consequence of this, if the long time (steady state) roughness scales as $w(t \rightarrow \infty) \sim L^\zeta$, then the conventional relation $\zeta = z\beta$ is replaced by

$$\zeta = z_v \beta_v - \tfrac{1}{2}. \tag{2}$$

With the additional result [14,17,20]

$$z_v = 1 + \zeta, \tag{3}$$

one can obtain β_v in terms of ζ. Both Eqs. (2) and (3) differ from renormalization group results [3] for an interface driven with a constant applied force. In particular, the RG analysis yields *no* simple relation between the dynamic exponent z and the roughness exponent ζ, unlike Eq. (3). Nor is there any reason to expect the anomalous scaling form of Eq. (1), and therefore $\zeta = z\beta$ should hold instead of Eq. (2).

We first demonstrate that, despite appearances, the RG flows are unaffected by going from a constant force driving mode to a constant velocity driving mode. The standard equation for zero-temperature driven dynamics of an interface is [21]

$$\partial_t h(x,t) = \nabla^2 h(x,t) + Y(h(x,t);x) + F. \tag{4}$$

Here $\nabla^2 h(x,t)$ comes from the (short-ranged) elastic energy of the interface, $Y(h;x)$ is a pinning force that comes from a random (impurity) potential, and F is the external driving force. (Of course, it is not necessary that all lattice growth models—e.g., the Sneppen model [13]—can be mapped to this or any continuum equation.) We have neglected Kardav-Parisi-Zhang-like terms [22,23]. Using the Martin-Siggia-Rose method [24] and introducing an auxiliary field $\hat{h}(x,t)$, one can construct [3] a generating functional Z that can be written as

$$Z = \int [dh][d\hat{h}] \exp\Bigg[\int d^d x\, dt\, i\hat{h}(x,t)$$
$$\times \{\partial_t h - \nabla^2 h - F - Y(h;x)\}\Bigg], \tag{5}$$

where integrating out the auxiliary field yields a product of δ functions that imposes Eq. (4). If instead the interface is driven at constant velocity, Eq.(5) is replaced by

$$Z' = \int [dF][d\mu][dh][d\hat{h}] \exp\Bigg[\int d^d x\, dt\, i\hat{h}(x,t)$$
$$\times \{\partial_t h - \nabla^2 h - F(t) - Y(h;x)\} + i\mu(t)\{v - \partial_t h\}\Bigg]. \tag{6}$$

This extension is actually not difficult to understand: at any time t, by first integrating over $\mu(t)$ and $\hat{h}(x,t)$, we obtain δ-function constraints that impose $\langle \partial_t h \rangle = v$ (the average here is a spatial average) and Eq. (5) with a (as yet unknown) driving force $F(t)$. Now performing the integral over F, the integral together with the first constraint sets $F(t)$ to be whatever it has to be for $\langle \partial_t h \rangle$ to be equal to v. One is left with the second constraint, i.e., Eq. (4), with $F(t)$ adjusted to ensure constant velocity. Even though $F(t)$ is now a dynamical variable instead of a parameter, since it is not a field, i.e., it has no spatial dependence [nor has $\mu(t)$], when short distance fluctuations are eliminated under renormalization there are no extra loop corrections in Z' compared to Z.

As mentioned earlier, even though the RG fixed point is the same for constant force and constant velocity driving, the behaviors of scaling functions are different for the two driving modes in the $\xi \gg L$ regime. To illustrate this, we first consider the scaling of the duration of an avalanche as a

function of its linear size. This has the general scaling form

$$t(l,L,\xi)=l^z T(l/L,\xi/L). \tag{7}$$

Here l is the linear extent of the avalanche (in the d transverse directions; in the direction the interface moves, the extent is $\sim l^\zeta$), L is the linear size of the system, ξ is the correlation length, and z is the nontrivial dynamical exponent from the RG. For a fixed l and ξ, as $L\to\infty$, the avalanche duration should be independent of L since the system is uncorrelated on length scales much bigger than ξ [25–27]. It should also be independent of ξ, since requiring a total growth rate of vL^d for the system does not affect any avalanche from proceeding with its own intrinsic time scale. Thus $T(0,0)$ is a nonzero constant.

In the other regime of $\xi\gg L$, i.e., if the $v\to0$ limit is taken *before* $L\to\infty$, the avalanches are nonoverlapping in time. In the constant velocity driving mode, where the velocity constraint is imposed, any single avalanche proceeds at a fixed rate. Therefore, for the constant velocity driving mode, $t(l,L,\xi)$ must be inversely proportional to the rate at which the avalanche is allowed to proceed, which is vL^d. Requiring that t should have such a dependence on v and L, using the result [3] $v\sim\xi^{\zeta-z}$, it is straightforward to verify that

$$\lim_{\xi/L\to\infty} t(l,L,\xi)\sim l^{\zeta+d}/(vL^d). \tag{8}$$

This yields an apparent dynamical exponent of $z_v=\zeta+d$, which is the $(d+1)$-dimensional generalization of Eq. (3). Note that there is *no* change in the apparent dynamical exponent in the $\xi\gg L$ regime for constant force driving, where an avalanche is allowed to proceed at its intrinsic rate (through parallel updating of lattice sites). This is why the RG estimate for z for two dimensional CDWs agrees with the exact result for Abelian sandpiles [12].

The analysis of the interface roughness for a system of linear size L, $w(t,L)$, proceeds in a similar manner. At a velocity v, the interface roughness has the scaling form

$$w(t,L,\xi)=L^\zeta W(t/L^z;\xi/L). \tag{9}$$

For fixed t and ξ, the roughness must have a well defined $L\to\infty$ limit:

$$\lim_{L\to\infty} w(t,L,\xi)=\xi^\zeta W_1(t/\xi^z). \tag{10}$$

For large t, the roughness saturates to the steady state form $w\sim\xi^\zeta$. For $\xi\to\infty$ (the large L limit has been taken first), or equivalently for small t, the roughness is ξ independent, i.e., $w\sim t^{\zeta/z}$.

In the other regime of $\xi\gg L$, the apparent exponents are once again different for constant velocity driving. In this case, the dynamics are controlled by the imposed velocity, i.e., the v and t dependence of Eq. (9) occurs only in the combination $\tau=vL^d t$. Using $v\sim\xi^{\zeta-z}$, this implies

$$\lim_{\xi/L\to\infty} w(t,L,\xi)=L^\zeta W_2(\tau/L^{\zeta+d}). \tag{11}$$

(With extremal dynamics, where one site moves forward at every time step, $\tau=t$.) In the large time limit, the roughness approaches the steady state form $w\sim L^\zeta$, i.e., $W_2(\infty)$ is a

constant. It is not obvious how to extract the behavior of the function W_2 when its argument is small, but physical arguments supported by numerical results [14,17] show that $w(\tau,L)$ must have a residual $L^{-d/2}$ dependence. Therefore,

$$w\sim(vL^d t)^{\beta_v}/L^{-d/2} \tag{12}$$

with

$$\beta_v=\frac{\zeta+d/2}{\zeta+d}, \tag{13}$$

which is Eq. (10) of Ref. [17] (in d dimensions).

Equations (8) and (12) give the $v\to0$ scaling behavior of avalanche durations and interface roughness, respectively, for constant velocity driving. We note once again that these are for $\xi\gg L$, i.e., $v\to0$ before $L\to\infty$. As mentioned before, even in critical phenomena for equilibrium phase transitions, as one approaches the transition for a fixed system size, one sees a change in the scaling form of dynamical variables. However, it is generally possible to obtain the behavior in this regime by requiring that there should be no dependence on (say) the reduced temperature t for the behavior of a finite size system. This requirement comes from the fact that there are *no* thermodynamic singularities for a finite sized system: it is possible to go smoothly from one side of the phase transition to another. No such requirement exists for dynamical phase transitions, and one must be careful about possible v dependence, even in the $v\to0$ regime [2].

There are other mechanisms as well that can make the connection between RG exponents and apparent scaling difficult. For instance, for CDWs below the depinning threshold, the periodicity of the dynamical variable (the CDW phase) prevents it from advancing by more than 2π anywhere in a single avalanche, but the same periodicity makes a region that has just avalanched susceptible to an imminent ''retriggering'' of a fresh avalanche [28]. The low frequency dynamics thus ''sees'' a nontrivial analog of the roughness exponent, and the distribution of avalanche sizes is singular as $v\to0$. Another example is for interface roughness itself, where the steady state roughness over a subsystem of size x in a system with size L scales as $w^2(x;L)\sim x^2 L^{2\zeta-2}$ when $\zeta>1$ [29]. This is because the Fourier transform of the steady state roughness must be well behaved as $L\to\infty$, and for $\zeta>1$ $w^2(x;L)$ is dominated by long wavelength modes with $q\sim1/L$.

In this paper we have shown that RG flows for depinning transitions are the same whether the system is driven with a constant force or a constant velocity. However, in the critical regime, the apparent scaling behavior of physical quantities can be different for the two cases: ''hidden'' powers of v and L contribute to scaling dimensions in Eqs. (8) and (12). Scaling functions have the same *form*, but can have different *limits* for the two driving modes. This is unlike what is seen in equilibrium critical phenomena, and seems to be more common in the constant velocity driving mode. Despite this caveat, one can obtain results for one driving mode from the other.

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